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Bubbly flow and its relation to conduction in composites

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Following Wallis, the relation between non-viscous bubbly flow and conduction in composites is examined. The bubbles are treated as incompressible and correspond to non-conducting inclusions. A simple relation is found between the effective conductivity and the energy coefficient which is agreement with previous calculations. It is shown that the energy coefficient is frame dependent and, in the frame of zero volumetric flux, is equal to the virtual mass density. Zuber's virtual mass density corresponds to the conductivity of the Hashin–Shtrikman coated-sphere geometry. This connection is exploited to extend Zuber's result to ellipsoidal bubbles. The hyperbolicity of effective equations derived from a variational principle is analysed for various bubble configurations. Without bubble clustering the equations are ill-posed (unstable). However, when the bubbles group into ellipsoidal clusters the resulting effective equations are well-posed for a wide range of parameter values.

1. Introduction

The effective conductivity of composite materials consisting of spheres suspended in a matrix has been studied extensively over the last century. Difficulties concerning conditionally convergent series and integrals arise when attempting to compute effective conductivity. Different techniques have been devised to overcome these problems such as Batchelor's renormalization (Jeffrey 1973) or Ewald summation (see for example, Smith & Ashcroft 1988).

Similar difficulties arise when trying to deduce effective properties in bubbly flows. One such effective quantity is the virtual mass coefficient of a bubbly mixture. The virtual mass density is a phenomenological function that depends on the void fraction and bubble configuration. Loosely speaking, it is a measure of how much kinetic energy in the liquid is due to bubble motion. In some models of bubbly flow another phenomenological function, related to the virtual mass, plays a key role in determining the stability of the effective equations. More explicitly, the question of hyperbolicity in the dilute limit rests on the coefficients of the α and α^2 terms of this function. Here α is the void fraction. See for example Lhuillier (1985), Geurst (1985), Wallis (1989*b*), or Pauchon & Smereka (1991). This indicates that accurate calculation of these coefficients is decidedly more important than a small correction to the effective equations.

The virtual mass of a spherical bubble in an inviscid, irrotational, unbounded liquid is

$$m = \tfrac{1}{2}\rho_{\ell}\,\tau, \tag{1.1}$$

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where ρ_ℓ is the density of the liquid and τ is the volume of the bubble. The virtual mass density for a bubbly flow is a non-trivial problem because one must consider the mutual interaction of the bubbles. In order to reduce these interactions dilute dispersions of bubbles are usually considered. Results in the literature have been expressed in the form

$$m = \tfrac{1}{2}\rho_\ell \alpha(1+b\alpha)+O(\alpha^3), \tag{1.2}$$

where α is the void fraction and b is a constant that depends on the details of the pairwise interaction of the bubbles.

An attempt to calculate the virtual mass density was made by Zuber (1964) who allowed for the presence of the other bubbles by surrounding each sphere with another sphere of a radius equal to the average interbubble distance. The outer sphere was considered to be stationary. He found $b = 3$.

Another derivation of the virtual mass density was performed by van Wijngaarden (1976). He considered a cloud of bubbles initially at rest and calculated the average velocity attained by the bubbles under a quick acceleration of the liquid and found $b = 2.78$. Several technical difficulties arise in this calculation and are overcome using results from Batchelor (1972) and Jeffrey (1973).

In a subsequent development, Bieshuevel & Spoelstra (1989) extended the relationship between the fluid impulse and dipole strength of a single sphere to a collection of bubbles. They also encountered difficulties due to conditionally convergent integrals which are also overcome using Batchelor's renormalization technique. They found $b = 3.32$.

Wallis (1989*b*) showed that one can use the effective conductivity (electrical or thermal) to derive the added mass and the kinetic energy due to the relative motion of the bubbles. Using Maxwell's expression for the conductivity he showed that one could recover Zuber's result for the added mass. Wallis then showed that there is another definition of the added mass and explains how it is related to Zuber's result. Wallis' study is important because it allows one to use known effective conductivities thereby avoiding many technical difficulties. Earlier than Wallis, Mercadier (1981) had made some important observations concerning effective conductivity and Zuber's expression for the virtual mass.

In this paper we continue Wallis' discussion and further explore the analogue between effective conductivity and bubbly flow. We present an alternative derivation of Wallis' results and suggest that there are other natural choices of reference frames that would give different virtual mass coefficients. By considering two different frames we recover Wallis' results and the formulae of van Wijngaarden (1976), Kok (1988) and Biesheuvel & Spoelstra (1989). It is also shown that Zuber's (1964) formula is exact for the coated-sphere geometry of Hashin & Shtrikman (1962). Zuber's result is then generalized to bubbly flows containing oblate ellipsoidal bubbles.

In §4 we investigate the relationship between the effective conductivity and another phenomenological constant used in the literature. This constant, sometimes mistaken for the virtual mass density, relates the square of the velocity fluctuations in the liquid phase to the relative velocity squared. Wallis (1989*b*) calls this constant the exertia and we call it the Reynolds stress coefficient. This constant is crucial in applying a variational principle to deduce effective equations describing the mixture. Also the specific form of this constant as a function of void fraction has been shown to determine the stability and hyperbolicity of the effective equations.

The effective conductivity of different distributions of bubbles is discussed in §6 and the form of the Reynolds stress coefficient is investigated. One important conclusion is that the effective equations derived from a variational principle are not hyperbolic for any unclustered bubbly fluid. However, we give arguments to suggest that the formation of ellipsoidal bubble clumps yields hyperbolic effective equations.

2. The analogy with effective conductivity

The physical problem that motivates the following discussion is uniform bubbly flow in a pipe or duct. In this situation it is observed that the bubbles are, on average, moving in the same direction. Furthermore we assume that in any section of pipe long enough to contain a large number of bubbles they are all moving with roughly the same velocity. This is consistent with the assumptions used in averaging bubbly flow, see for example, Biesheuvel & van Wijngaarden (1984), Geurst (1985), or Wallis (1989*a*).

As an idealization of this flow we first consider a finite collection of spherical bubbles contained in a square box of fluid denoted V. Next we periodically extend this to fill all space. The fluid will be considered to be inviscid, incompressible and irrotational. Each bubble is assumed rigid and given the same velocity denoted by $\boldsymbol{U}$. Since the flow is irrotational, the velocity field is given by

$$\boldsymbol{u}_\ell = -\nabla\phi \quad \text{where} \quad \nabla^2\phi = 0, \tag{2.1}$$

and the boundary condition on each spherical bubble surface is

$$\nabla\phi\cdot\hat{\boldsymbol{n}}_K = -\boldsymbol{U}\cdot\hat{\boldsymbol{n}}_K, \tag{2.2}$$

where ϕ is the velocity potential and $\hat{\boldsymbol{n}}_K$ is the outward unit normal of the Kth bubble. We choose our frame of reference to be one where ϕ is periodic on V. The implications of this choice will be discussed later.

The kinetic energy per unit volume of the liquid is

$$K = \frac{\rho_\ell}{V}\int_{V_\ell} \tfrac{1}{2}|\nabla\phi|^2\,\mathrm{d}V, \tag{2.3}$$

where V_ℓ is the portion of V that is occupied by liquid. Now since ϕ depends linearly on U then K must be proportional to U^2. We write

$$K = \tfrac{1}{2}\kappa U^2. \tag{2.4}$$

This equation defines what we will call the energy coefficient, κ, of the bubbles. It reduces to the same definition of the virtual mass as for a single bubble, see Milne-Thomson (1968). The goal is to find the kinetic energy for a given arrangement of bubbles and thereby deduce the energy coefficient for the collection.

This is achieved by making a comparison with the problem of heat (or electrical) conductivity through a matrix of unit conductivity dispersed with insulating spheres. Once the comparison is made the virtual mass will be linked to the effective conductivity of the mixture.

We begin by introducing a new field.

$$T = \boldsymbol{U}\cdot\boldsymbol{r}+\phi. \tag{2.5}$$

From (2.2) the boundary condition for T on the bubble surface is

$$\nabla T\cdot\hat{\boldsymbol{n}}_K = 0. \tag{2.6}$$

Since $\nabla\phi\rightarrow 0$ for small bubble concentration this implies $\nabla T\rightarrow \boldsymbol{U}$ in the above limit. The analogy is now clear. The above situation corresponds to insulating spheres in a conducting matrix with an imposed temperature gradient $\boldsymbol{U}$. For convenience we take the conductivity of the continuous phase to be one.

The effective conductivity σ^* of such a medium is defined as the constant of proportionality linking the average heat flux with the average temperature gradient

$$\frac{1}{V}\int_V \sigma(\boldsymbol{r})\,\nabla T(\boldsymbol{r})\,\mathrm{d}V = \sigma^*\frac{1}{V}\int_V \nabla T(\boldsymbol{r})\,\mathrm{d}V, \tag{2.7}$$

where $$\sigma(\boldsymbol{r}) = \begin{cases} 0 & \text{for} \quad \boldsymbol{r} \text{ in } V_s \\ 1 & \text{for} \quad \boldsymbol{r} \text{ in } V_\ell \end{cases}$$

is the thermal conductivity, σ^* is the effective conductivity†; V_s and V_ℓ are the volumes that contain bubbles and liquid respectively. Since ϕ is a periodic function on V with mean zero (2.5) implies

$$\frac{1}{V}\int_V \nabla T\,\mathrm{d}V = \boldsymbol{U}. \tag{2.8}$$

Using Green's theorem the definition (2.7) for the effective conductivity can be replaced by the equivalent definition, see for example Bergman (1978),

$$\frac{1}{V}\int_V \sigma(\nabla T)^2\,\mathrm{d}V = \sigma^* U^2. \tag{2.9}$$

Since $\sigma = 0$ in V_s we have

$$\frac{1}{V}\int_{V_\ell} (\nabla T)^2\,\mathrm{d}V = \sigma^* U^2. \tag{2.10}$$

Now substituting (2.5) into (2.10) we have

$$\frac{1}{V}\int_{V_\ell} (\nabla\phi)^2\,\mathrm{d}V = (\sigma^* - 1 + \alpha)\,U^2 - 2\boldsymbol{U}\cdot\frac{1}{V}\int_{V_\ell} \nabla\phi\,\mathrm{d}V, \tag{2.11}$$

where α is the void fraction. Using (2.5), (2.7) and (2.8) we have

$$\frac{1}{V}\int_{V_\ell} \nabla\phi\,\mathrm{d}V = (\sigma^* + \alpha - 1)\,\boldsymbol{U}. \tag{2.12}$$

This combined with (2.11) gives

$$\frac{1}{V}\int_{V_\ell} (\nabla\phi)^2\,\mathrm{d}V = (1 - \sigma^* - \alpha)\,U^2. \tag{2.13}$$

Comparing (2.13) with (2.3) and (2.4) we have for the energy coefficient

$$\kappa(\alpha) = \rho_\ell(1 - \sigma^* - \alpha). \tag{2.14}$$

This result was obtained by Wallis (1989*b*) (see his §3.3) using the same analogy but with a different approach, where σ^* is $1/\beta$ in that paper. It is interesting to notice that from (2.12) we have

$$\frac{\rho_\ell}{V}\int_{V_\ell} \boldsymbol{u}_\ell\,\mathrm{d}V = \kappa(\alpha)\,\boldsymbol{U}, \tag{2.15}$$

† Strictly speaking σ^* should, of course, be treated as a tensor. However, we will consider media which are isotropic or have axial symmetry with the applied field in the axial direction. In this case σ^* represents an eigenvalue of the conductivity matrix and $\boldsymbol{U}$ is an eigenvector.

which indicates that $\kappa(\alpha)$ is also the constant of proportionality between the induced liquid momentum and the bubble velocity.

Although we have defined the results for periodic extensions of a particular configuration in V we can let V become infinite and it is physically clear that the results hold for random statistically homogeneous configurations as well. Therefore we conclude that the formula for the energy coefficient holds for any configuration of bubbles for which an effective conductivity is defined. Furthermore, the analysis extends to bubbles of any shape provided that the effective conductivity is known and we assume the shape of the bubbles remains fixed. By assuming that the bubbles have a fixed shape we can neglect surface tension.

In a subsequent section we shall study the relationship between bubble distribution and virtual mass but for now we recall Jeffrey's (1973) calculation for the effective conductivity of a random array of spherical insulators in matrix of unit conductivity

$$\sigma^* = 1-\tfrac{3}{2}\alpha+k\alpha^2+O(\alpha^3), \tag{2.16}$$

where $k = 0.75$ for well separated suspensions and $k = 0.59$ for well mixed suspensions. Substitution of (2.16) into (2.14) produces

$$\kappa(\alpha) = \rho_\ell\left(\frac{\alpha}{2}-k\alpha^2\right)+O(\alpha^3). \tag{2.17}$$

This result is significantly different from the formulae for the virtual mass coefficient obtained by Zuber (1964), van Wijngaarden (1976), and Biesheuvel & Spoelstra (1989) who all find k close to -1.5 instead of 0.59 or 0.75. This might suggest that the definition of energy coefficient and the virtual mass coefficient are different for collections of bubbles while being the same for single bubbles. Also, it is important to realize that the calculation done by these investigators is done in the zero-volume-flux frame of reference. In the next section we will show that by transforming to this frame of reference we can recover these results.

3. Virtual mass and frame of reference

We now transform to new frame of reference moving with velocity λU relative to the frame in which ϕ is periodic. The bubble velocity in this frame is

$$\boldsymbol{U}' = (1-\lambda)\,\boldsymbol{U}, \tag{3.1}$$

and the liquid velocity is

$$\boldsymbol{u}_\ell' = -\lambda \boldsymbol{U}-\boldsymbol{\nabla}\phi, \tag{3.2}$$

where the prime denotes transformed variables. The kinetic energy density is

$$K' = \frac{\rho_\ell}{V}\int_{V_\ell}\tfrac{1}{2}|\boldsymbol{u}_\ell'|^2\,\mathrm{d}V. \tag{3.3}$$

Substituting (3.2) into (3.3) and using (2.12) and (2.13) we find

$$K' = \tfrac{1}{2}\rho_\ell[(1-\alpha)\,(1-\lambda)^2+\sigma^*(2\lambda-1)]\,U^2. \tag{3.4}$$

Also, the energy coefficient is the constant relating the transformed bubble velocity to the transformed kinetic energy. Therefore it follows that

$$\kappa'(\alpha) = \rho_\ell\left[1-\alpha+\frac{\sigma^*(2\lambda-1)}{(1-\lambda)^2}\right]. \tag{3.5}$$

Next, we shall compute the value of λ needed to give zero volumetric flux. In the moving frame we have

$$j_0' = \alpha U' + (1-\alpha)\langle u_\ell' \rangle_\ell, \tag{3.6}$$

where j_0' is the volumetric flux, U' is the bubble velocity and $\langle \cdot \rangle_\ell$ denotes an average over the liquid phase. Using (3.2) and (2.12) one can show that

$$\langle u_\ell' \rangle_\ell = -\lambda U + \left(1 - \frac{\sigma^*}{1-\alpha}\right) U. \tag{3.7}$$

This combined with (3.6) yields

$$j_0' = [-\lambda + 1 - \sigma^*]\, U. \tag{3.8}$$

Now $j_0' = 0$ holds if

$$\lambda = 1 - \sigma^*. \tag{3.9}$$

Substituting (3.9) into (3.5) produces

$$\kappa_0(\alpha) = \rho_\ell \left(\frac{1}{\sigma^*} - 1 - \alpha\right), \tag{3.10}$$

where $\kappa_0(\alpha)$ is the energy coefficient in the zero-volumetric-flux frame. Wallis (1989*b*) also obtains (3.10) using a different approach. An inspection of (3.7) and (3.4) reveals that there is no analogy to (2.15) for any $\lambda \neq 0$. This was noticed by Wallis (1989*b*).

Both van Wijngaarden (1976) and Biesheuvel & Spoelstra (1989) consider dilute, random, and well-mixed suspensions. The effective conductivity according to (2.15) is given by

$$\sigma^* = 1 - \tfrac{3}{2}\alpha + 0.59\alpha^2 + O(\alpha^3). \tag{3.11}$$

Substituting (3.11) into (3.10) produces

$$\kappa_0(\alpha) = \tfrac{1}{2}\rho_\ell\, \alpha(1 + 3.32\alpha) + O(\alpha^3). \tag{3.12}$$

If we compare (3.12) with the result for the virtual mass coefficient of Biesheuvel & Spoelstra (1989) we find it is identical, see (1.2). Also (3.10) is in agreement with Biesheuvel & Spoelstra's calculation of the virtual mass coefficient for the periodic lattice.

Next we examine the calculation of Zuber (1964). He considered the effects of the other bubbles by surrounding each bubble with a 'security' bubble. The security bubble has zero velocity corresponding to the zero-volumetric-flux frame. The entire space is then filled with these security bubble elements. Since spheres do not tile space well a fractal arrangement is required to fill up the volume. See figure 1. This is the same configuration as considered by Hashin & Shtrikman (1962) in their coated-sphere geometry. They show that the effective conductivity for insulating spheres can be calculated exactly for all volume fractions, α, and is

$$\sigma^* = \frac{1-\alpha}{1+\frac{1}{2}\alpha}. \tag{3.13}$$

This expression for the conductivity was obtained by Maxwell (1881) for random distributions and the derivation is only accurate to $O(\alpha)$ in that case. Substituting (3.13) into (3.10) gives

$$\kappa_0(\alpha) = \rho_\ell \frac{\alpha}{2}\left(\frac{1+2\alpha}{1-\alpha}\right). \tag{3.14}$$

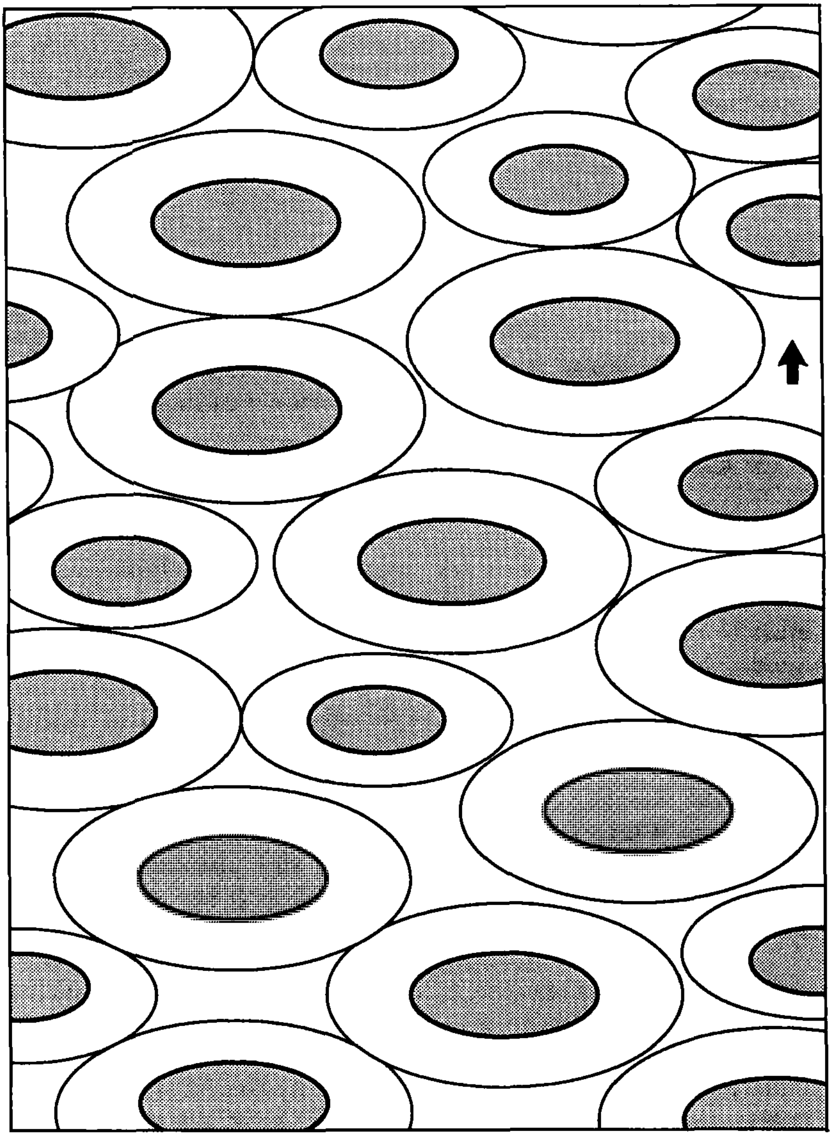

FIGURE 1. This is the coated-ellipsoidal geometry for oblate ellipsoids. The thick lines are the bubble surfaces and the thin lines are the security ellipsoids. For spheres this is the configuration which gives Zuber's result exactly. The direction of flow is marked by the arrow.

This is precisely Zuber's result for the virtual mass coefficient. Therefore we conclude that Zuber's result is exact for the geometry of bubbles pictured in figure 1 when those bubbles are spheres.

These results demonstrate that the virtual mass coefficient is equal to the energy coefficient in the zero-drift-flux frame. Furthermore it suggests that had these authors done their calculations in different frame they would have obtained different results. This raises the question: What is the correct frame for defining the virtual mass, if any? The choice of reference frame (zero drift flux) made by van Wijngaarden (1976) and Biesheuvel & Spoelstra (1989) is important because it allows them to remove difficulties with conditionally convergent integrals. However, the observation of Wallis (1989*b*) that (2.13) and (2.15) have the same coefficient suggests that the periodic reference frame could also be a natural choice. Certainly, another natural choice is the frame where the average liquid velocity is zero. This becomes more evident in the following section: in §4 we shall discuss another phenomenological function that is frame indifferent and appears naturally in some bubbly flow models.

4. Reynolds stresses and hyperbolicity

When the bubbles move through the liquid they induce both a mean flow, $\boldsymbol{u}_\ell$, and a fluctuating part, $\boldsymbol{u}'_\ell$. By analogy with turbulence the product of the fluctuations is called the Reynolds stress tensor. The volume average of the Reynolds stress tensor plays a key role in bubbly fluids. The importance of this term will be outlined in what follows.

The average kinetic energy of the liquid phase can be written as

$$K = \tfrac{1}{2}\rho_\ell(1-\alpha)\left(\langle \boldsymbol{u}_\ell\rangle_\ell^2 + \langle \boldsymbol{u}_\ell'^2\rangle_\ell\right), \tag{4.1}$$

where $\boldsymbol{u}'_\ell = \boldsymbol{u}_\ell - \langle \boldsymbol{u}_\ell\rangle_\ell$, and $\langle\cdot\rangle_\ell$ denotes an average over the liquid volume. If it is assumed that the flow is irrotational then the deviation of the liquid velocity from its mean must be the result of the relative motion of the bubbles with respect to the average liquid velocity. To express this notion one writes

$$\langle \boldsymbol{u}_\ell'^2\rangle_\ell = f(\alpha)\,\boldsymbol{u}_\mathrm{r}^2, \tag{4.2}$$

where $\boldsymbol{u}_\mathrm{r} = \boldsymbol{U} - \langle \boldsymbol{u}_\ell\rangle_\ell$ and $f(\alpha)$ is a phenomenological function that depends on the configuration of the bubbles. This relationship has been used to Lhuillier (1985) and Pauchon & Smereka (1991); Geurst (1985) uses a similar relationship except that $f(\alpha)$ is replaced by $m_G(\alpha)/(1-\alpha)$, where $m_G(\alpha)$ is the phenomenological function introduced by Geurst and should not be confused with the virtual mass density. Wallis (1989*b*) has discussed the importance of this function in modelling bubbly flow.

One possible way to derive effective equations for bubbly flow is to substitute (4.2) into (4.1) and interpret the result as an energy functional. One can then apply a variational principle where the averaged mass conservation laws act as constraints. Effective equations have been derived in this way by Geurst (1985) and Pauchon & Smereka (1991). Geurst's equations were examined by Wallis (1989*a*).

An examination of the effective equations derived in this manner reveals that $f(\alpha)$ plays the deciding role in the determination of the hyperbolicity of the model. More specifically, for the case of massless bubbles, Pauchon & Smereka (1991) define the following function:

$$\Gamma(\alpha) = \frac{1}{1-\alpha} + \frac{f(\alpha)}{\alpha^2(1-\alpha)}, \tag{4.3}$$

and that show if

$$2\Gamma^2 - \Gamma\Gamma'' \geqslant 0 \tag{4.4}$$

then the effective equations are hyperbolic and the solutions are stable. Here the prime denotes differentiation with respect to α. Equation (4.4) is equivalent to (6.1) of Geurst (1985) in the limit of incompressible and massless bubbles; see Appendix. Hyperbolicity is important for otherwise the equations are ill-posed. If we let

$$f(\alpha) = c_1\alpha + c_2\alpha^2 + O(\alpha^3), \tag{4.5}$$

then in the dilute limit the stability condition reduces to

$$1+c_1+c_2 < 0. \tag{4.6}$$

In what follows we shall use results from §2 to derive $f(\alpha)$ in terms of the effective conductivity. Our first observation is that $\boldsymbol{U}$ is not the relative velocity of the bubbles since the bubbles tend to induce an average relative velocity in the liquid. Indeed, from (2.14) we infer

$$\langle \boldsymbol{u}_\ell \rangle_\ell = \langle -\nabla\phi \rangle_\ell = \left(1-\frac{\sigma^*}{1-\alpha}\right)\boldsymbol{U}, \tag{4.7}$$

which implies

$$\boldsymbol{u}_\mathrm{r} = \boldsymbol{U} - \langle \boldsymbol{u}_\ell \rangle_\ell = \frac{\sigma^*}{1-\alpha}\boldsymbol{U}. \tag{4.8}$$

Next we note that

$$\begin{aligned}\langle \boldsymbol{u}_\ell'^2 \rangle_\ell &= \langle \boldsymbol{u}_\ell^2 \rangle_\ell - \langle \boldsymbol{u}_\ell \rangle_\ell^2 \\ &= \langle |\nabla\phi|^2 \rangle_\ell - \langle \nabla\phi \rangle_\ell^2.\end{aligned} \tag{4.9}$$

Using (2.13) and (4.7) in (4.9) reveals that

$$\langle \boldsymbol{u}_\ell'^2 \rangle_\ell = \frac{\sigma^*}{1-\alpha}\left(1-\frac{\sigma^*}{1-\alpha}\right)U^2. \tag{4.10}$$

Combining (4.10) and (4.8) shows that

$$\langle \boldsymbol{u}_\ell'^2 \rangle_\ell = \frac{1-\alpha}{\sigma^*}\left(1-\frac{\sigma^*}{1-\alpha}\right)\boldsymbol{u}_\mathrm{r}^2. \tag{4.11}$$

Therefore we have

$$f(\alpha) = \frac{1-\alpha}{\sigma^*} - 1. \tag{4.12}$$

Wallis (1989*b*) also obtained (4.12) and he calls $f(\alpha)$ the exertia. This result can also be obtained from the energy coefficient formula (3.5) once it is recognized that $(1-\alpha)\rho_\ell f(\alpha)$ is the energy coefficient in the frame where the average fluid velocity is zero, corresponding to $\lambda = 1-\sigma^*/(1-\alpha)$.

If we consider well-mixed random suspensions then we have $\sigma^* = 1-\frac{3}{2}\alpha+0.59\alpha^2+O(\alpha^3)$. Substitution of this into (4.12) and expanding about $\alpha = 0$ gives

$$f(\alpha) = \tfrac{1}{2}\alpha + 0.16\alpha^2 + O(\alpha^3). \tag{4.13}$$

Kok (1988) also examined the form of $f(\alpha)$. A comparison shows that $f(\alpha) = -\alpha^2+(1-\alpha)\,k(\alpha)$ where $k(\alpha)$ is defined by equation (19) of Kok's paper. Kok shows that for random well-mixed bubbles $k(\alpha) = \frac{1}{2}\alpha(1+3.32\alpha)+O(\alpha^3)$, which is in agreement with (4.13).

By comparing (4.13) with (4.6) we conclude that the effective equations for bubbly flow derived by Geurst (1985) and Pauchon & Smereka (1991) are ill-posed. Using a different approach both Lhuillier (1985) and van Beek (1982) arrived at a similar conclusion. van Beek suggested that the ill-posedness is physical and not an inadequacy of the model. He associated the ill-posedness with the clustering tendency of the bubbles. Geurst (1985) suggests that bubbles should tend to line up with the mean flow. He argues that this results in non-isotropic bubbly flows which will be hyperbolic. With the connection between σ^* and $f(\alpha)$ established it should be much easier to investigate these possibilities. We shall present some preliminary results in §6.

5. Bubbly flows with elliptical bubbles

Larger bubbles rising in liquids tend to be closer to oblate ellipsoids rather than spheres. In view of this let us consider a natural generalization of Zuber's model where the spheres are replaced by ellipses. The conductivity for this geometry has been examined by Milton (1981), Bergman (1982) and Tartar (1985). The basic formulae below are found in Bergman's article.

We consider a mixture of oblate ellipsoidal bubbles where each bubble is surrounded by a security ellipsoid, see figure 1. The ellipsoids are confocal and their eccentricities are e_1 and e_2 respectively. The eccentricities are related by

$$\alpha = \frac{1+e_1^2}{e_1^3}\frac{e_2^3}{1+e_2^2}, \tag{5.1}$$

where α is the void fraction. The conductivity of the mixture (when the matrix has unit conductivity and the bubbles are insulating) is

$$\sigma^* = 1-\frac{\alpha}{1-N}, \tag{5.2}$$

where $N = n_1 - \alpha n_2$ and

$$n_i(e_i) = \left(1+\frac{1}{e_i^2}\right)\left(1-\frac{1}{e_i}\tan^{-1} e_i\right), \quad i = 1,2.$$

Substituting (4.2) into (3.10) we find that the energy coefficient in the zero-volumetric-flux frame is

$$\kappa_0(\alpha) = \rho_\ell \frac{\alpha(N+\alpha)}{1-\alpha-N}. \tag{5.3}$$

Expanding (5.3) in α for $e_1 \ll 1$ we have

$$\frac{\kappa_0(\alpha)}{\rho_\ell} = J\alpha + \tfrac{2}{3}(J+1)^2\alpha^2 + O(\alpha^3), \tag{5.4}$$

where
$$J = \frac{n_1}{1-n_1} = \frac{\tan^{-1}(e_1)-e_1}{\tan^{-1} e_1 - (e_1^2+1)^{-1} e_1}.$$

This expression for J may also be expressed in terms of r, the ratio of the major axis to the minor axis since $e_1 = (r^2-1)^{\frac{1}{2}}$. We have

$$J = \frac{(r^2-1)^{\frac{1}{2}} - \cos^{-1}(r^{-1})}{\cos^{-1}(r^{-1}) - (r^2-1)^{\frac{1}{2}} r^{-2}}.$$

From (4.12) and (5.2) the Reynolds stress function is

$$f(\alpha) = \frac{\alpha N}{1-\alpha-N}. \tag{5.5}$$

The condition (4.4) for hyperbolicity is somewhat complex but simplifies if we consider only small void fractions with $e_1 \ll 1$. For the Reynolds stress function we have

$$f(\alpha) = J\alpha + \tfrac{1}{3}(2J-1)(J+1)\alpha^2 + O(\alpha^3). \tag{5.6}$$

Comparing (5.6) with (4.6) we conclude that oblate ellipsoidal bubbly flow is unstable within this approximation.

6. The stability of various other bubble arrangements

It was seen that the virtual mass and the Reynolds stress function could both be simply expressed in terms of the effective conductivity using (4.12). The behaviour of these two phenomenological functions for dilute, random, and isotropic distributions was discussed in §§2 and 4. In this section we shall exploit the relationship with effective conductivity to analyse several other bubble configurations.

Let us now consider a periodic array of bubbles. This is interesting, not because we would expected to see such arrangements, but because it will provide an exact solution to the dilute case and an approximate solution for the non-dilute random case. The periodic case was first studied by Rayleigh (1892) and has been subsequently studied by many others. One important result of this work is that for cubic arrays of insulating spheres in a matrix of unit conductivity the effective conductivity is

$$\sigma^* = \frac{1-\alpha}{1+\frac{1}{2}\alpha} + O(\alpha^{\frac{10}{3}}). \tag{6.1}$$

The exact nature of the higher-order terms depends on the nature of the cubic lattice (simple cubic, body centred or face centred), see McPhedran & McKenzie (1978) or Sangani & Acrivos (1983) for examples. Not surprisingly, the leading-order term in (6.1) is exactly the same as Hashin's coated sphere geometry. Furthermore Sangani & Acrivos (1983) show for the random close packed case ($\alpha_{\max} = 0.62$) that $\sigma^* \approx 0.27$. This suggests that for insulating spheres (3.13) is a good approximation over the complete range of α. Although bubbly flows are not known to exist for high void fraction, (3.13) combined with the results of §§3 and 4 could be useful for fluidized beds when inertial effects are important.

Next we consider anisotropic distributions. Geurst (1985) conjectured that (4.6) would be satisfied because the bubbles would tend to line up with the mean flow and would result in the coefficient of the α^2 term in (4.5) being less than $-\frac{3}{2}$, since $c_1 = \frac{1}{2}$ for spherical bubbles.

The simplest anisotropic configuration imaginable is a rectangular lattice of spheres. Geurst's hypothesis would entail the lattice spacing being smaller in the direction of the applied field (streamwise direction) compared to the cross-stream direction. The conductivity for such a model can be inferred from the work of Sangani & Acrivos (1983) although they only studied cubic lattices. They show that the effective conductivity takes the form

$$\sigma^* = 1 + \frac{4\pi}{\tau_0} A_{00}, \tag{6.2a}$$

where τ_0 is the volume of the unit cell (here a rectangle) and for insulating spheres A_{00} is given by

$$-\frac{a^3}{A_{00}} = -\frac{1}{2} + \frac{4\pi a^3}{3\tau_0} + O(a^{10}), \tag{6.2b}$$

where a is the bubble radius. The exact details of the higher-order terms are determined by the details of the geometry: see Sangani & Acrivos (1983) equations (27) and (29). Note that (6.2*b*) combined with (6.2*a*) gives (6.1).

The important feature of (6.2) is that the leading-order terms are a function only of the volume of the unit cell not its shape. Substituting (6.1) into (4.12) gives

$$f(\alpha) = \tfrac{1}{2}\alpha + O(\alpha^{\frac{10}{3}}),$$

and we conclude that dilute periodic bubble mixtures all have approximately the same virtual mass and Reynolds stress coefficient. The upshot of this is that the effective equations for dilute isotropic and anisotropic flows with no clustering of the bubbles suffer the same fate, they are ill-posed.

Next we will draw on some ideas from composite material research to investigate the properties of bubbly flow where the bubbles are arranged into clumps. As a simplified model we consider the coated sphere geometry studied by Hashin & Shtrikman (1962) that was generalized to ellipsoids by Milton (1981) and Bergman (1982). We imagine a clump of bubbles to be an (oblate or prolate) ellipsoid filled with both bubbles and liquid surrounded by another (oblate or prolate) ellipsoid of pure liquid. The ellipsoids are confocal with eccentricities e_1 and e_2 respectively. We then let the entire mixture be filled with those composite ellipsoids. See figure 2. Now we let p represent the volume fraction of the mixture that is occupied by the bubble-filled ellipsoids and set β to be the void fraction within these ellipsoids due to the bubbles. The void fraction of the entire mixture is then

$$\alpha = p\beta, \tag{6.3}$$

and the effective conductivity in the direction of the mean flow is

$$\sigma^* = 1 - \frac{p}{(1-\sigma_\beta)^{-1} - (n_1 - pn_2)}, \tag{6.4}$$

where σ_β is the effective conductivity of the bubble mixture inside the inner ellipsoid. For oblate (disk-shaped) ellipsoids n_1 and n_2 are given by (5.2) and the relationship between e_1, e_2, and p given by (5.1) (with α replaced by p). For prolate (cigar-shaped ellipsoids we have

$$n_i(e_i) = \frac{1-e_i^2}{2e_i^3}\left(\ln\left(\frac{1+e_i}{1-e_i}\right) - 2e_i\right), \quad i = 1, 2, \tag{6.5}$$

and
$$p = \frac{1-e_1^2}{e_1^3}\frac{e_2^3}{1-e_2^2}. \tag{6.6}$$

To investigate the stability we must first compute $f(\alpha)$. We shall use (3.13) as a good approximation of σ_β (with α replaced by β, of course). Substituting (6.4) into (4.12) gives

$$f(\alpha) = \frac{\alpha(1-\beta[1-3(n_1-pn_2)])}{2-3\alpha+\beta[1-3(n_1-pn_2)]}. \tag{6.7}$$

To continue we need to specify more information about p and β. In this study we shall restrict our analysis to only two cases. We first take p to be independent of α (and β is then determined by (6.3)) and second we take β to be independent of α (and p is determined by (6.3)).

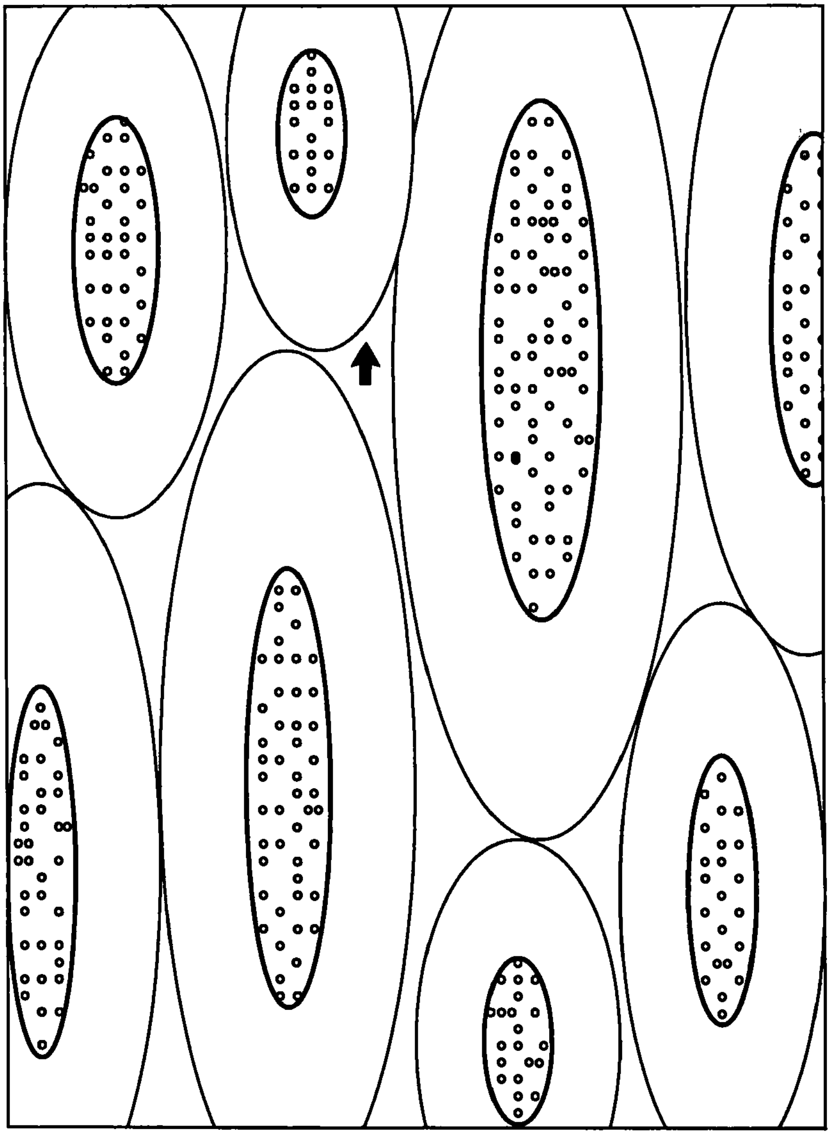

FIGURE 2. The idealized clump arrangement is shown for the prolate case. The thick-lined ellipsoids are filled with spherical bubbles and have a void fraction β. The thin-lined ellipsoids are the security ellipsoids which contain pure liquid. The direction of flow is marked by the arrow.

Case 1. p is independent of α.

Here $\beta = \alpha/p$ and (6.5) becomes

$$f(\alpha) = \frac{\alpha(1-(\alpha/p)[1-3(n_1-pn_2)])}{2-3\alpha+(\alpha/p)[1-3(n_1-pn_2)]}. \tag{6.8}$$

Now since p is a constant then it follows that n_1-pn_2 is also a constant. If we substitute (6.8) into (4.3) then the hyperbolicity condition (4.4) becomes

$$\frac{6p^2}{\alpha^3}\frac{(3p+3(n_1-pn_2)-1)}{(\alpha[3p+3(n_1-pn_2)-1]-2p)^3} \geqslant 0. \tag{6.9}$$

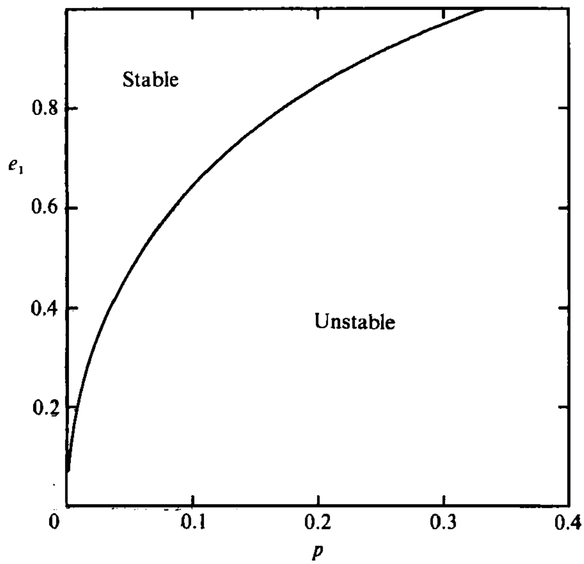

FIGURE 3. This shows the hyperbolicity condition for the prolate ellipsoids for the case when p is independent of the void fraction. This curve is independent of the void fraction.

Bergman (1982) shows

$$\tfrac{1}{3}(1-p) \leqslant n_1 - pn_2 \leqslant 1-p \quad \text{for the oblate case,} \tag{6.10a}$$

$$0 \leqslant n_1 - pn_2 \leqslant \tfrac{1}{3}(1-p) \quad \text{for the prolate case.} \tag{6.10b}$$

With these inequalities one can easily show that the denominator of (6.9) is always negative for both oblate and prolate ellipsoids. For oblate ellipsoids (6.10a) implies that the numerator of (6.9) is always positive. This indicates that the hyperbolicity condition can never be satisfied for the oblate geometry. For the prolate geometry the hyperbolicity condition (6.9) is satisfied if

$$p + n_1 - pn_2 \leqslant \tfrac{1}{3}. \tag{6.11}$$

Since e_2 is related to e_1 and p through (6.6) then (6.11) is an inequality involving the two independent variables e_1 and p. In the prolate geometry we have $0 \leqslant e_1 \leqslant 1$, and $e_1 \rightarrow 0$ indicates that the bubble clumps become spherical. For $e_1 \rightarrow 1$ the bubble clumps become cylinders. It is important to notice that (6.11) is independent of α. The condition (6.11) is displayed in figure 3. The results show that as volume fraction of clumps, p, increases their eccentricity must also increase if (6.11) is to be satisfied.

Case 2. β is independent of α.

Here $p = \alpha/\beta$ and the expression for $f(\alpha)$ becomes

$$f(\alpha) = \frac{\alpha(1-\beta[1-3(n_1-(\alpha/\beta)\,n_2)])}{2-3\alpha+\beta[1+3(n_1-(\alpha/\beta)\,n_2)]}. \tag{6.12}$$

This is more complex than before because n_2 is a function of α because e_2 depends on $p = \alpha/\beta$ as seen in (6.6). The hyperbolicity condition is then found by substituting (6.12) into (4.3) and examining (4.4). Here we have two independent variables, e_1 and β. Since it seems impossible to write down a simple analytical expression for the stability condition we have computed it numerically. For the prolate ellipsoids the hyperbolicity condition could never be satisfied. However, in contrast to the previous

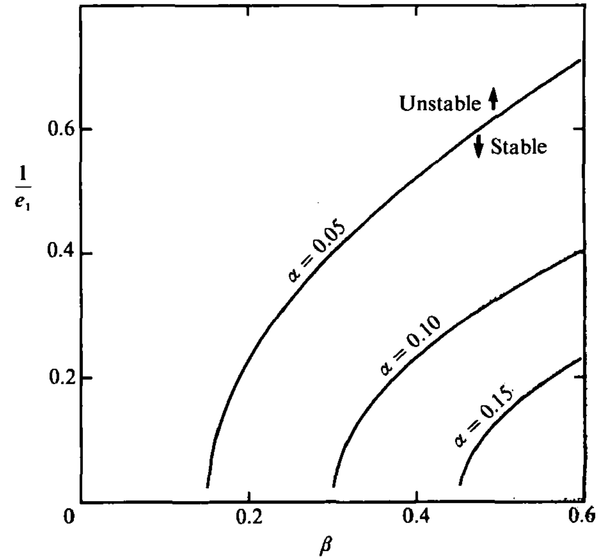

FIGURE 4. This shows the hyperbolicity condition for the oblate ellipsoids for the case when β is independent of the void fraction.

case the oblate ellipsoid geometry did satisfy the hyperbolicity condition under a wide range of circumstances. The results of the numerical study are summarized in figure 4 where it was found convenient to use $1/e_1$ instead of e_1. Also we have restricted $\beta < 0.62$, the maximum void fraction for random arrangements of spheres. For the prolate ellipsoids we have $0 \leqslant e_1 \leqslant \infty$, and $e_1 \to 0$ indicates that the bubble clumps become spherical, while $e_1 \to \infty$ means that the bubble clumps tend to horizontal slabs. By horizontal, we mean perpendicular to the direction of flow.

The results contained in figure 4 show that for a fixed void fraction, α, that void fraction inside the clump, β, must be sufficiently high for the hyperbolicity condition to be satisfied. It also shows that bubble clumps tending to slabs, $(1/e_1) \to 0$, tend to be more stable. An important feature displayed in figure 4 is that as the void fraction, α, is increased the region of hyperbolicity shrinks and that for $\alpha \gtrsim 0.2$ there is no hyperbolic region. This is interesting because experiments show that bubbly flow cannot exist above a critical void fraction. Experimentally the critical void fraction is in the range 0.25 to 0.45, in rough agreement with our analysis.

Other mechanisms may also be important in determining the hyperbolicity of the bubbly fluid flow. In particular Biesheuvel & Gorissen (1990) have found that liquid viscosity and diffusive effects resulting from chaotic bubbly motion are relevant for stability. Although we ignore such effects, our calculations could be useful in determining some of the phenomenological functions in the model of Biesheuvel & Gorissen, and would allow the inclusion of the effects of bubble clustering.

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Appendix

Here we show that Geurst's stability condition (see Geurst 1985, equation (6.1)) is identical with (4.4) when one considers massless and incompressible bubbles. Our starting point is equation (6.1) of Geurst's paper. We have:

$$D = -16\left(\frac{RT}{M}\right)^5\left[1+\gamma\frac{1-\alpha}{\alpha}+\frac{m_G(\alpha)}{1-\alpha}\frac{1}{\alpha^2}\right]^3\left[1+\left(1+\frac{1}{\gamma}\frac{1-\alpha}{\alpha}\right)\frac{m_G(\alpha)}{1-\alpha}\right]^{-5}$$
$$\times\left\{\left[1+\gamma\frac{1-\alpha}{\alpha}+\frac{m_G(\alpha)}{1-\alpha}\frac{1}{\alpha^2}\right]\left[1+3\frac{m_G(\alpha)}{1-\alpha}+2m_G'(\alpha)+\tfrac{1}{2}(1-\alpha)\,m_G''(\alpha)\right]\right.$$
$$\left.-\left[1-\frac{1-3\alpha}{\alpha^2(1-\alpha)}m_G(\alpha)+\frac{m_G'(\alpha)}{\alpha}\right]^2\right\}w_0^2+O(w_0^4). \tag{A 1}$$

where R is the ideal gas constant, T is the temperature, M is the molecular mass, γ is the ratio of the gas density to the liquid density, w_0 is the relative velocity of the bubbles to the liquid, and $m_G(\alpha)$ is the phenomenological function that Geurst uses in the kinetic energy. The stability condition is given as $D>0$. Now if we take the limit as $\gamma \rightarrow 0$ of (A 1) and use $m_G(\alpha) = (1-\alpha)f(\alpha)$ then (A 1) becomes

$$D = -16\left(\frac{p_g\,\alpha}{\rho_\ell(1-\alpha)f(\alpha)}\right)^5\left[1+\frac{f(\alpha)}{\alpha^2}\right]^3 G(\alpha)\,w_0^2, \tag{A 2}$$

where we have used the ideal gas law and ignored terms of $O(w_0^4)$ (as does Geurst). Here p_g is the absolute gas pressure and

$$G(\alpha) = \left[1+\frac{f(\alpha)}{\alpha^2}\right][1+f(\alpha)+(1-\alpha)f'(\alpha)+\tfrac{1}{2}(1-\alpha)^2f''(\alpha)]$$
$$-\left[1+\frac{(2\alpha-1)f(\alpha)}{\alpha^2}+\frac{1-\alpha}{\alpha}f'(\alpha)\right]^2.$$

Since the factor in front of $G(\alpha)$ in (A 2) is always negative then the stability condition becomes $G(\alpha)<0$. By straightforward manipulations (best done with symbolic computation) one can establish that

$$\frac{2G(\alpha)}{\alpha^2(1-\alpha)^4} = -2\Gamma'^2+\Gamma\Gamma'',$$

where $\Gamma(\alpha)$ is given by (4.3). Therefore we see that (4.4) is equivalent to Geurst's stability condition.

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